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S-duality Invariant Variables and Actions from the two Scalars of the type IIB Superstring Theory

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Abstract

Using two scalar fields of the type IIB superstring theory, i.e. the dilaton (the NS-NS scalar) and axion (the R-R scalar), we construct a number of scalars, differential forms and various 10-dimensional actions, which are invariant under the S-duality transformations. The invariant scalars help us to generalize the invariant actions. The generalized actions can be added to the bosonic part of the action of the type IIB supergravity and hence obtain a most extended action with the S-duality symmetry.

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1 Introduction

Study of the low-energy string effective actions has shown itself to be most profitable. Deformation and generalization of the supergravity actions have been studied from the various point of view. Among them the modified versions of the type IIB supergravity are more desirable [1, 2, 3]. This is due to the fact that the type IIB supergravity contributes to the AdS/CFT duality [4]. In addition, it contains the axion (the R-R scalar field) and dilaton (the NS-NS scalar field), which are corresponding to the D-instanton and the Hodge duals of their field strengths are associated with the D7-brane and NS7-brane. Study of the axion and dilaton fields may also shed light on the F-theory [5], which has the type IIB superstring as an underlying theory.

On the other hand there is the $SL(2, \mathbf{R})$ group which is certainly a symmetry group of the type IIB supergravity theory (e.g. see the Refs. [6, 7, 8, 9]). The type IIB superstring theory is conjectured to be $SL(2, \mathbf{Z})$ duality invariant. There is also an important special case, i.e. the S-duality, which relates strong and weak coupling phases of a given theory in some cases, whereas in some other situations strong and weak coupling regimes of two different theories are connected.

In this paper, from the dilaton and axion fields and their field strengths and also the Hodge duals of their field strengths we construct various S-duality invariant differential r-forms with $r = 0, 1, 2, 10$. These differential forms enable us to obtain six independent 10-dimensional actions which separately have the S-duality symmetry. By using the invariant scalars these actions can be generalized such that the generalized versions of them also to be S-duality symmetric. The initial actions or extended versions of them can be added to the bosonic part of the action of the type IIB supergravity to obtain a most generalized theory which possesses the S-duality invariant. In fact, many other S-duality invariant scalars, differential forms and actions can be made which are not independent of the previous ones. They will be detached. Note that in this article, from the word "*S-duality*" we mean a duality which is generated by the $\mathbf{Z_2}$-subgroup of the group $SL(2, \mathbf{R})$.

This paper is organized as follows. In Sec. 2, the S-duality invariant scalars will be constructed. In Sec. 3, the S-duality invariant differential forms will be made. In Sec. 4, by using the above scalars and differential forms, various S-duality invariant actions will be established. In Sec. 5, independent invariants will be detached from the dependent ones. Section 6 is devoted to the conclusions.

2 Invariant scalars

We recall that the bosonic massless excitations of the type IIB theory consist of the graviton $G_{\mu\nu}$, dilaton Φ and antisymmetric tensor $B_{\mu\nu}$ in the NS-NS sector, and the R-R counterparts are the axion $C_0 \equiv C$, another antisymmetric tensor field $C_{(2)\mu\nu}$ and a four-index antisymmetric potential $C_{(4)\mu\nu\rho\lambda}$ with self Hodge-dual field strength. In addition, in both sectors the Hodge duality also procreates some other form fields.

The S-duality is generated by the $\mathbf{Z_2}$-subgroup of the group $SL(2,\mathbf{R})$. This determines S-duality transformations of the scalar fields C and Φ as $\tau \to \tau' = -1/\tau$ where $\tau = C + ie^{-\Phi}$ is the axion-dilaton modulus. This duality mixes the two scalars as in the following

$$\begin{aligned} e^{-\Phi'} &= \frac{e^{-\Phi}}{C^2 + e^{-2\Phi}} \ , \\ C' &= -\frac{C}{C^2 + e^{-2\Phi}} \ . \end{aligned} \tag{1}$$

Since the string coupling is given by $g_s = e^{\Phi}$, when the axion vanishes, the first equation relates the weak and strong coupling regimes of the type IIB superstring theory. The Einstein metric, which will be used, is an S-duality invariant variable

$$G'_{\mathrm{E}\mu\nu} = G_{\mathrm{E}\mu\nu}, \tag{2}$$

where the Einstein frame and string frame metrics are related by $G_{\mathrm{E}\mu\nu} = e^{-\Phi/2} G_{\mu\nu}$. From now on we utilize the word "*invariant*" instead of "*S-duality invariant*", which means invariance under the transformations (1) and (2).

According to the Eqs. (1) we acquire the following equations

$$\begin{aligned} \frac{e^{-2\Phi'}}{C'^2 + e^{-2\Phi'}} &= \frac{e^{-2\Phi}}{C^2 + e^{-2\Phi}} \equiv \sigma_1(C,\Phi) \ , \\ \frac{C'^2}{C'^2 + e^{-2\Phi'}} &= \frac{C^2}{C^2 + e^{-2\Phi}} \equiv \sigma_2(C,\Phi) \ . \end{aligned} \tag{3}$$

These equations represent two invariant scalars σ_1 and σ_2, i.e. $\sigma_i(C',\Phi') = \sigma_i(C,\Phi)$ for $i = 1, 2$.

Since twice S-dualization leaves C and Φ invariant, i.e. $(C')' = C$ and $(\Phi')' = \Phi$, we can construct more invariant scalars from C and Φ. These new scalars have the common feature

$$\sigma_3(C,\Phi) = f(C,\Phi) + f(C',\Phi') + g(C,\Phi) g(C',\Phi'), \tag{4}$$

where C' and Φ' should be replaced from the Eqs. (1). The functionals f and g are arbitrary but well-behaved. For the next purposes we consider them to be dimensionless.

The invariance of the scalar field σ_3 is manifest, that is, the first and the second terms of the right-hand-side get exchange while the third term transforms to itself and hence $\sigma_3(C', \Phi') = \sigma_3(C, \Phi)$.

Various combinations of the invariant scalars σ_1, σ_2 and σ_3 define new invariant scalars. For example, various powers of them, i.e. $\{(\sigma_1)^{n_1}, (\sigma_2)^{n_2}, (\sigma_3)^{n_3} | n_i \in R \,, i = 1, 2, 3\}$ are new invariant scalars. Generally, any functional of these scalars reveals an invariant scalar. We shall apply such functionals.

Besides, there are also the following invariant scalars which comprise derivatives of the fields Φ and C,

$$e^{2\Phi'} F'_\mu F'^\mu + H'_\mu H'^\mu = e^{2\Phi} F_\mu F^\mu + H_\mu H^\mu, \tag{5}$$

$$e^{2\Phi'}[2C' F'_\mu H'^\mu + (C'^2 - e^{-2\Phi'}) H'_\mu H'^\mu] = e^{2\Phi}[2C F_\mu H^\mu + (C^2 - e^{-2\Phi}) H_\mu H^\mu], \tag{6}$$

$$e^{4\Phi'}[(C'^2 - e^{-2\Phi'}) F'_\mu F'^\mu - 2C' e^{-2\Phi'} F'_\mu H'^\mu] = e^{4\Phi}[(C^2 - e^{-2\Phi}) F_\mu F^\mu - 2C e^{-2\Phi} F_\mu H^\mu], \tag{7}$$

where $H_\mu = \partial_\mu \Phi$, $F_\mu = \partial_\mu C$, $H'_\mu = \partial_\mu \Phi'$ and $F'_\mu = \partial_\mu C'$. The indices are raised by the Einstein metric, e.g. $F^\mu = G_{\rm E}^{\mu\nu} F_\nu$. Some other kinds of the invariant scalars can be seen in the integrands of the actions (23), (31), (32) and in the Eq. (37). Furthermore, invariant scalars with higher order derivatives also can be constructed which are not appropriate for making invariant actions.

3 Invariant differential forms

Exterior derivatives of the scalar fields give 1-forms. Combination of these 1-forms and their Hodge duals determines some higher order invariant differential forms which will be used to obtain several invariant actions.

• **Invariant 1-forms**

By exterior derivative of the scalar (4) we acquire the following invariant 1-form

$$\Omega \equiv \Big(\frac{\partial f}{\partial C} + g' \frac{\partial g}{\partial C}\Big) F + \Big(\frac{\partial f}{\partial \Phi} + g' \frac{\partial g}{\partial \Phi}\Big) H + \Big(\frac{\partial f'}{\partial C'} + g \frac{\partial g'}{\partial C'}\Big) F' + \Big(\frac{\partial f'}{\partial \Phi'} + g \frac{\partial g'}{\partial \Phi'}\Big) H', \tag{8}$$

where $\Omega = d\sigma_3$, $H = d\Phi$, $F = dC$, $H' = d\Phi'$, $F' = dC'$, $f' \equiv f(C', \Phi')$ and $g' \equiv g(C', \Phi')$. In this feature the self-duality of Ω is manifest, i.e. $\Omega(C', \Phi'; F', H') = \Omega(C, \Phi; F, H)$. However, for the next purposes we write it in the form

$$\Omega = \alpha(C, \Phi) F + \beta(C, \Phi) H, \tag{9}$$

where the functions $\alpha(C,\Phi)$ and $\beta(C,\Phi)$ have the definitions

$$\alpha(C,\Phi) \equiv \frac{\partial f}{\partial C} + \frac{\partial f'}{\partial C'}\frac{\partial C'}{\partial C} + \frac{\partial f'}{\partial \Phi'}\frac{\partial \Phi'}{\partial C} + g'\frac{\partial g}{\partial C} + g\Big(\frac{\partial g'}{\partial C'}\frac{\partial C'}{\partial C} + \frac{\partial g'}{\partial \Phi'}\frac{\partial \Phi'}{\partial C}\Big),$$
$$\beta(C,\Phi) \equiv \frac{\partial f}{\partial \Phi} + \frac{\partial f'}{\partial C'}\frac{\partial C'}{\partial \Phi} + \frac{\partial f'}{\partial \Phi'}\frac{\partial \Phi'}{\partial \Phi} + g'\frac{\partial g}{\partial \Phi} + g\Big(\frac{\partial g'}{\partial C'}\frac{\partial C'}{\partial \Phi} + \frac{\partial g'}{\partial \Phi'}\frac{\partial \Phi'}{\partial \Phi}\Big).$$

Note that these functions are not invariant scalars. There is also an anti-invariant 1-form in the Eq. (38).

• **Invariant 2-form**

Exterior derivative of the Eqs. (1) indicates the 1-forms

$$H' = \frac{1}{C^2 + e^{-2\Phi}}\Big(2CF + (C^2 - e^{-2\Phi})H\Big),$$
$$F' = \frac{1}{(C^2 + e^{-2\Phi})^2}\Big((C^2 - e^{-2\Phi})F - 2Ce^{-2\Phi}H\Big). \tag{10}$$

Wedge product of these forms specifies the following 2-form

$$e^{\Phi'}F' \wedge H' = e^{\Phi}F \wedge H \equiv \mathcal{F}. \tag{11}$$

Thus, the 2-form $\mathcal{F}$ is self S-dual. This equation elaborates that $\mathcal{F}$ is field strength of the 1-form $A = Ce^{\Phi}H$, and or $A' = C'e^{\Phi'}H'$. These 1-forms are related via the gauge transformation $A' = A + \Lambda$ where Λ is a closed 1-form. We shall investigate this.

• **Invariant 10-forms**

We know that Hodge duality definition of a differential form is characterized by the metric of the manifold. According to this, for the next purposes, we define this duality via the Einstein metric. For example, the Hodge dual of the 1-form F, which is a 9-form $\widetilde{F}$, has the components

$$\widetilde{F}_{\mu_1\mu_2\cdots\mu_9} = \sqrt{-G_{\rm E}}F_{\mu}G_{\rm E}^{\mu\nu}\varepsilon_{\nu\mu_1\mu_2\cdots\mu_9}, \tag{12}$$

where $G_{\rm E} = \det G_{{\rm E}\mu\nu}$. The Levi-Civita symbol $\varepsilon_{\mu_1\mu_2\cdots\mu_{10}}$ has the components ± 1 and 0. Therefore, the Hodge duality on the Eqs. (9) and (10) exhibits the 9-forms

$$\widetilde{\Omega} = \alpha(C,\Phi)\widetilde{F} + \beta(C,\Phi)\tilde{H}, \tag{13}$$

$$\tilde{H}' = \frac{1}{C^2 + e^{-2\Phi}}\Big(2C\widetilde{F} + (C^2 - e^{-2\Phi})\tilde{H}\Big),$$
$$\widetilde{F}' = \frac{1}{(C^2 + e^{-2\Phi})^2}\Big((C^2 - e^{-2\Phi})\widetilde{F} - 2Ce^{-2\Phi}\tilde{H}\Big). \tag{14}$$

The Hodge dual of the Eq. (8) explicitly reveals that the 9-form $\widetilde{\Omega}$ is self S-dual.

Let us give a brief description regarding the 9-forms $\widetilde{F}$ and $\tilde{H}$. Two different local combinations of these forms accompanied by other form fields have been associated with the D7-brane and NS7-brane. The NS7-brane is related to the D7-brane by the S-duality, where in these duality transformations the effective actions of these branes are related to each other [10]. These transformations also imply that the D7-brane and NS7-brane do not form a doublet under the S-duality. It has been commonly accepted that there are bound states of p D7-branes and q NS7-branes which transform as doublets. However, in the Ref. [11] it has been elucidated that the 7-brane states transform as triplets, and the corresponding supergravity solutions have been studied in it.

Combining (9) with (13) and also (10) with (14) via the wedge product specifies the following invariant 10-forms

$$\Omega \wedge \widetilde{\Omega} = [\alpha(C,\Phi)F + \beta(C,\Phi)H] \wedge [\alpha(C,\Phi)\widetilde{F} + \beta(C,\Phi)\tilde{H}], \tag{15}$$

$$e^{2\Phi'} F' \wedge \widetilde{F}' + H' \wedge \tilde{H}' = e^{2\Phi} F \wedge \widetilde{F} + H \wedge \tilde{H}. \tag{16}$$

$$e^{2\Phi'}[2C'F' \wedge \tilde{H}' + (C'^2 - e^{-2\Phi'})H' \wedge \tilde{H}'] = e^{2\Phi}[2CF \wedge \tilde{H} + (C^2 - e^{-2\Phi})H \wedge \tilde{H}], \tag{17}$$

$$e^{4\Phi'}[(C'^2 - e^{-2\Phi'})F' \wedge \widetilde{F}' - 2C'e^{-2\Phi'}H' \wedge \widetilde{F}'] = e^{4\Phi}[(C^2 - e^{-2\Phi})F \wedge \widetilde{F} - 2Ce^{-2\Phi}H \wedge \widetilde{F}]. \tag{18}$$

These 10-forms have been precisely written on the basis of the Einstein frame. Note that the invariant 10-form $e^{\Phi}(H \wedge \widetilde{F} - F \wedge \tilde{H})$ is zero.

By introducing various functions of the invariant scalars $\{\sigma_i | i = 1,2,3\}$ into the invariant differential forms (9), (11), (13), (15)-(18) we obtain a set of the extended invariant differential forms. For example, for an arbitrary function f the generalized 2-form $f(\sigma_1, \sigma_2, \sigma_3)e^{\Phi} F \wedge H$ is invariant.

4 Invariant actions

4.1 The main invariant actions

Making use of the 10-forms (15)-(18) we establish the following actions, which have the S-duality symmetry

$$S_1 = -\frac{1}{9!4\kappa^2} \int_{\mathcal{M}} e^{-2\Phi}[\alpha(C,\Phi)F + \beta(C,\Phi)H] \wedge [\alpha(C,\Phi)\widetilde{F}_{\mathrm{s}} + \beta(C,\Phi)\tilde{H}_{\mathrm{s}}], \tag{19}$$

$$S_2 = -\frac{1}{9!4\kappa^2}\int_{\mathcal{M}}(F\wedge\widetilde{F}_{\rm s} + e^{-2\Phi}H\wedge\tilde{H}_{\rm s}), \tag{20}$$

$$S_3 = -\frac{1}{9!4\kappa^2}\int_{\mathcal{M}}[2CF\wedge\tilde{H}_{\rm s} + (C^2 - e^{-2\Phi})H\wedge\tilde{H}_{\rm s}], \tag{21}$$

$$S_4 = -\frac{1}{9!4\kappa^2}\int_{\mathcal{M}}e^{2\Phi}[(C^2 - e^{-2\Phi})F\wedge\widetilde{F}_{\rm s} - 2Ce^{-2\Phi}H\wedge\widetilde{F}_{\rm s}], \tag{22}$$

where κ is the 10-dimensional gravitational constant and $\mathcal{M}$ indicates the spacetime manifold. These actions have been written in the string frame which is more usual. The 9-forms in the two frames are related by $\widetilde{F} = e^{-2\Phi}\widetilde{F}_{\rm s}$ and $\tilde{H} = e^{-2\Phi}\tilde{H}_{\rm s}$. The subscript "s" refers to the string frame.

These actions also take the following features

$$S_1 = -\frac{1}{4\kappa^2}\int_{\mathcal{M}}d^{10}x\sqrt{-G}e^{-2\Phi}[\alpha(C,\Phi)F_\mu + \beta(C,\Phi)H_\mu][\alpha(C,\Phi)F^\mu_{\rm s} + \beta(C,\Phi)H^\mu_{\rm s}], \tag{23}$$

$$S_2 = -\frac{1}{4\kappa^2}\int_{\mathcal{M}}d^{10}x\sqrt{-G}(F_\mu F^\mu_{\rm s} + e^{-2\Phi}H_\mu H^\mu_{\rm s}), \tag{24}$$

$$S_3 = -\frac{1}{4\kappa^2}\int_{\mathcal{M}}d^{10}x\sqrt{-G}[2CF_\mu H^\mu_{\rm s} + (C^2 - e^{-2\Phi})H_\mu H^\mu_{\rm s}], \tag{25}$$

$$S_4 = -\frac{1}{4\kappa^2}\int_{\mathcal{M}}d^{10}x\sqrt{-G}e^{2\Phi}[(C^2 - e^{-2\Phi})F_\mu F^\mu_{\rm s} - 2Ce^{-2\Phi}F_\mu H^\mu_{\rm s}], \tag{26}$$

where $G = \det G_{\mu\nu}$, and $F^\mu_{\rm s}$ and $H^\mu_{\rm s}$ refer to the string frame, i.e. $F^\mu_{\rm s} = G^{\mu\nu}F_\nu$ and $H^\mu_{\rm s} = G^{\mu\nu}H_\nu$. The action S_2 in the Einstein frame can be written in the form

$$S_2 = -\frac{1}{4\kappa^2}\int_{\mathcal{M}}d^{10}x\sqrt{-G_{\rm E}}\,\frac{\partial_\mu\tau\partial^\mu\bar{\tau}}{({\rm Im}\tau)^2}. \tag{27}$$

This clarifies that S_2 is a part of the action of the type IIB supergravity, which we found an alternative derivation for it.

The 2-form (11) has the components

$$\mathcal{F}_{\mu\nu} = e^\Phi(F_\mu H_\nu - F_\nu H_\mu). \tag{28}$$

This indicates a field strength $\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ where the $U(1)$ vector field A_μ is specified by

$$A_\mu = C\partial_\mu e^\Phi. \tag{29}$$

We observe that under the S-duality transformations (1) the tensor $\mathcal{F}_{\mu\nu}$ remains invariant. In addition, the 1-form $A = A_\mu dx^\mu$ transforms to $A' = A + \Lambda$ where Λ is the following closed 1-form

$$\Lambda = -\frac{2C^2 e^{\Phi}}{C^2 + e^{-2\Phi}}(F + CH). \tag{30}$$

Thus, we are lead to introduce the invariant action

$$S_5 = -\frac{1}{4g_{10}^2}\int_{\mathcal{M}} d^{10}x\sqrt{-G}e^{-3\Phi/2}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rm s}^{\mu\nu}, \tag{31}$$

where g_{10} is the 10-dimensional Yang-Mills coupling constant and $\mathcal{F}_{\rm s}^{\mu\nu} = G^{\mu\rho}G^{\nu\lambda}\mathcal{F}_{\rho\lambda} = e^{-\Phi}\mathcal{F}_{\rm E}^{\mu\nu}$. From this point of view the S-duality transformations (1) can be interpreted as gauge transformations.

In fact, under the transformation $C \to C + 1$, which induces $\tau \to \tau + 1$, the actions S_2 and S_5 are symmetric. This transformation accompanied by invariance under $\tau \to -1/\tau$ demonstrate that these actions possess the $SL(2;\mathbf{Z})$ symmetry.

Beside the action (31), advent of the invariant 2-form (11) inspires another invariant action

$$S_6 = -\frac{1}{4k^2}\int_{\mathcal{M}} d^{10}x\sqrt{-G}e^{-3\Phi/2}\mathcal{F}_{\mu\nu}f_{\rm s}^{\mu\nu}, \tag{32}$$

where k is an appropriate coupling constant and $f_{\mu\nu}$ is an antisymmetric tensor field. The nonzero elements of this tensor are arbitrary well-behaved functions of the invariant scalars $\{\sigma_i | i = 1, 2, 3\}$. Since the tensor $f_{\mu\nu}$ does not consist derivatives of the scalar fields this action does not show a Yang-Mills theory.

Since the spacetime is dynamical we should also include the kinetic term of the spacetime metric, i.e. the Einstein-Hilbert action

$$S_{\rm EH} = \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^{10}x\sqrt{-G_{\rm E}}\; R_{\rm E}\ , \tag{33}$$

where the scalar curvature $R_{\rm E}$ is constructed from the Einstein metric $G_{{\rm E}\mu\nu}$. When we select a combination of the invariant actions $\{S_l | l = 1, 2, \cdots, 6\}$, or a combination of the generalized versions of them (see Eqs. (34) and (35)), the action $S_{\rm EH}$ should be added to that combination.

4.2 Generalized invariant actions

It is possible to modify the actions S_1-S_6 such that the resultant actions also possess the S-duality symmetry. That is, deformation of the integrands of these actions by various

functions of the invariant scalars σ_1, σ_2 and σ_3 leads to another set of the invariant actions. Explicitly, let $\{W_l(\sigma_1, \sigma_2, \sigma_3)|l = 1, 2, \cdots, 6\}$ be a set of deformation functions. Therefore, the generalized versions of the actions $\{S_l|l = 1, 2, \cdots, 6\}$ have the common feature

$$A_l = -\frac{1}{4k_l^2}\int_{\mathcal{M}} d^{10}x\sqrt{-G}W_l(\sigma_1, \sigma_2, \sigma_3)\mathcal{L}_l \ , \quad l = 1, 2, \cdots, 6, \tag{34}$$

where $\sqrt{-G}\mathcal{L}_l$ is the Lagrangian density associated with the action S_l, and $\{k_l = \kappa|l = 1, 2, 3, 4\}$, $k_5 = g_{10}$ and $k_6 = k$. Similar modification can also be imposed on the Einstein-Hilbert action $S_{\rm EH}$. Since for any set of the deformation functions $\{W_l|l = 1, 2, \cdots, 6\}$ there is a corresponding set of the invariant actions $\{A_l|l = 1, 2, \cdots, 6\}$ the set of the main actions $\{S_l|l = 1, 2, \cdots, 6\}$ can be called a "basis".

In addition to the descendant actions (34), the following invariant action can also be considered

$$A_7 = -\frac{1}{4\kappa^2}\int_{\mathcal{M}} d^{10}x\sqrt{-G}e^{-5\Phi/2}W_7(\sigma_1, \sigma_2, \sigma_3), \tag{35}$$

where the deformation function W_7 has modified cosmological constant of the 10-dimensional spacetime. Since this action does not have dynamical term we deduce that W_7 represents an extended potential. By adjusting the function $e^{-5\Phi/2}W_7$ we receive a demanded potential and the cosmological variable, through the dynamics of the scalar fields C and Φ, also evolves to a desirable value.

Note that introducing various functions of the invariant scalars (5)-(7) and $\Omega_\mu\Omega^\mu$ into the basis actions $\{S_l|l = 1, 2, \cdots, 6\}$, similar to the Eq. (34), generates a new set of the generalized invariant actions. Since these scalars and any non-trivial functions of them raise order of derivatives, we do not apply them.

The descendant actions (34) and (35) stimulate us to write an action including the bosonic part of the type IIB supergravity for obtaining a generalized theory which possesses the S-duality symmetry. Therefore, we receive the following extended action

$$\bar{S} = S_{\rm IIB(bos)} + \sum_{l=1}^{7} w_l A_l. \tag{36}$$

Advent of the weights $w_l \in \{0, 1\}$ selects a subset of the actions $\{A_l|l = 1, 2, \cdots, 7\}$. Extracting a demanded theory determines this subset and explicit forms of the functions $\{W_l|l = 1, 2, \cdots, 7\}$. By introducing functions of $\{\sigma_i|i = 1, 2, 3\}$ into the various parts of the bosonic action $S_{\rm IIB(bos)}$ we can ultimately acquire the most generalized theory with the S-duality symmetry.

5 Dependent invariants

Making use of the Eqs. (1) we received the invariant scalars (3) and (4). It is possible to construct many other (anti-)invariant scalars from the Eqs. (1). They are not independent of $\{\sigma_i | i = 1, 2, 3\}$. Consequently, the various differential forms and corresponding actions, which are made of them usually are not independent of the previous ones. Let us illustrate this fact by the anti-invariant scalar

$$C' e^{\Phi'} = -C e^{\Phi}. \tag{37}$$

Extracting this equation via the Eqs. (3) ensures us that it is dependent. Effect of exterior derivative on this equation or on each of the equations in (3) leads to the following independent anti-invariant 1-form

$$e^{\Phi'}(F' + C'H') = -e^{\Phi}(F + CH). \tag{38}$$

The components of this 1-form inspire an invariant action, i.e.,

$$I = -\frac{1}{4\kappa^2}\int_{\mathcal{M}} d^{10}x\sqrt{-G}(F_\mu + CH_\mu)(F^\mu_{\rm s} + CH^\mu_{\rm s}). \tag{39}$$

We observe that this specific action is not independent, that is, $I = S_2 + S_3$.

The Hodge dual of the Eq. (38) exhibits an independent anti-invariant 9-form

$$e^{\Phi'}(\widetilde{F}' + C'\tilde{H}') = -e^{\Phi}(\widetilde{F} + C\tilde{H}). \tag{40}$$

Wedge product of the Eqs. (38) and (40) defines the invariant 10-form

$$e^{2\Phi'}(F' + C'H') \wedge (\widetilde{F}' + C'\tilde{H}') = e^{2\Phi}(F + CH) \wedge (\widetilde{F} + C\tilde{H}). \tag{41}$$

By summing the Eqs. (16) and (17) this 10-form is reproduced and hence loses its independence. The action associated with this 10-form also is not independent, i.e. it is given by summation of the Eqs. (20) and (21).

In addition to the indicated invariant scalars, the 9-forms $\widetilde{\Omega}$, $\widetilde{F}$, $\widetilde{F}'$, $\tilde{H}$ and $\tilde{H}'$ also enable us to obtain the following invariant scalars

$$\widetilde{\Omega} \cdot \widetilde{\Omega} = [\alpha(C,\Phi)\widetilde{F} + \beta(C,\Phi)\tilde{H}] \cdot [\alpha(C,\Phi)\widetilde{F} + \beta(C,\Phi)\tilde{H}], \tag{42}$$

$$e^{2\Phi'}\widetilde{F}' \cdot \widetilde{F}' + \tilde{H}' \cdot \tilde{H}' = e^{2\Phi}\widetilde{F} \cdot \widetilde{F} + \tilde{H} \cdot \tilde{H}, \tag{43}$$

$$e^{2\Phi'}[2C'\widetilde{F}' \cdot \tilde{H}' + (C'^2 - e^{-2\Phi'})\tilde{H}' \cdot \tilde{H}'] = e^{2\Phi}[2C\widetilde{F} \cdot \tilde{H} + (C^2 - e^{-2\Phi})\tilde{H} \cdot \tilde{H}], \tag{44}$$

$$e^{4\Phi'}[(C'^2 - e^{-2\Phi'})\widetilde{F}' \cdot \widetilde{F}' - 2C'e^{-2\Phi'}\widetilde{F}' \cdot \tilde{H}'] = e^{4\Phi}[(C^2 - e^{-2\Phi})\widetilde{F} \cdot \widetilde{F} - 2Ce^{-2\Phi}\widetilde{F} \cdot \tilde{H}], \quad (45)$$

where dot product between any two 9-forms $\widetilde{A}$ and $\widetilde{B}$ is defined by

$$\widetilde{A} \cdot \widetilde{B} = \widetilde{A}_{\mu_1 \cdots \mu_9} G_{\mathrm{E}}^{\mu_1 \nu_1} \cdots G_{\mathrm{E}}^{\mu_9 \nu_9} \widetilde{B}_{\nu_1 \cdots \nu_9}. \quad (46)$$

For our case these 9-forms are Hodge duals of the 1-forms A and B, and hence we have $\widetilde{A} \cdot \widetilde{B} = -9! G_{\mathrm{E}}^{\mu\nu} A_\mu B_\nu$. Thus, the above invariant scalars are not independent of the previous ones. The scalar (42), with the factor $\sqrt{-G_{\mathrm{E}}}/9!$, is the same as the integrand of the action (23), and the Eqs. (43)-(45) are equivalent to the Eqs. (5)-(7), respectively.

In fact, many other dependent invariant scalars, differential forms and actions can be made from the independent invariant variables. However, some of them maybe useful.

6 Conclusions

Two scalars of the type IIB superstring theory enabled us to establish various adequate S-duality invariant and anti-invariant quantities such as scalars, differential forms and actions. By combining these variables we received a large number of generalized invariants. Some of these new invariants are not independent of the previous ones.

The invariant actions $\{S_l | l = 1, 2, \cdots, 6\}$ form a basis for constructing many other invariant actions. That is, by introducing functions of the invariant scalars $\{\sigma_i | i = 1, 2, 3\}$ into the integrands of these actions we constructed generalized invariant actions. We also obtained some invariant scalars which contain derivatives of scalar fields. They are not appropriate to modify the basis actions. However, in the same fashion, if we modify each term of the bosonic action of the type IIB supergravity the resultant theory possesses the S-duality symmetry. Sum of all modified actions and the (modified) bosonic action of the type IIB supergravity specify a master theory which is symmetric under the S-duality transformations. This incorporated theory comprises various undetermined functions of the invariant scalars which can be adjusted to receive a demanded theory.

We observed that the invariant 2-form can be interpreted as field strength of an Abelian gauge field. Therefore, we were guided to a $U(1)$ Yang-Mills theory. The S-duality transformations exhibit the corresponding gauge transformation of the theory. In addition, from this invariant 2-form we obtained another invariant action which is different from the Yang-Mills theory.

Proliferation of the invariant variables demands its own subtlety. More independent invariants maybe exist. We illustrated that some of the invariants are not independent,

and hence the invariant actions which are built from them can be extracted from the basis actions. Note that by utilizing the self S-dual forms C_4 and $F_5 = dC_4 + \frac{1}{2}B \wedge dC_2 - \frac{1}{2}C_2 \wedge dB$ and combining them with the discussed invariant forms we can construct more invariant variables and actions.

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